Symmetries and metamorphoses[1]

Giuseppe Vitiello[2]

Quantum field dynamics may manifest itself in a variety of observable ordered patterns. The manifestations of the dynamical symmetry may be described in formal terms as metamorphoses. A crucial role in the metamorphosis processes is played by the coherence of the correlations generating order and self-similar fractal patterns. The local nature of the observations is at the origin of the dynamical rearrangement of symmetry generating the metamorphosis processes. The properties of dissipation, functional stability, the arising of the arrow of time are also discussed. Contrarily to what happens in a disordered system, the energy delivered to ordered patterns is distributed not only individually among the elementary constituents, but also to their coherent ordering correlations. Our conclusions apply to elementary particle physics, condensed matter physics, and to the physics of the living phase of the matter (biology and neuroscience). They can be as well applied to some aspects of linguistics in the generation of meanings in the transition from syntax to semantics.

1. Metamorphoses

The "question of metamorphosis" has always been object of attention in the study of natural phenomena, especially in phytology, zoology, and living systems in general. Notable examples are the studies by Linnaeus and Goethe scientist on the "process of plant growth" and transformation of the identical in which nature "unfolds the same dynamics." It is therefore interesting to see how in physics it is possible to speak of metamorphosis, morphology, transformation, and of the unfolding of the same dynamics in concrete formal terms that find confirmation in countless experimental findings.[3]

Some aspects of the formalism discussed below, typical of the physics of condensed matter and elementary particles, can also be usefully extended to the domain of living matter, from biology to neuroscience. In particular, the coherence of the microscopic dynamics, that is responsible of the generation of ordered structures, allows its manifestation at the macroscopic level and thus enables the extension of the of metamorphosis to all of nature.

2. Fields and their equations

Let me introduce some of the actors that appear in our story: the *fields*, their *equations*, their *transformations*.

Fields are well-defined mathematical quantities involving an infinite number of degrees of freedom and can be transformed according to certain precise prescriptions. If we observe for example the current of a fluid, it is unthinkable that we can measure the speed of each single molecule. We then introduce the *velocity field*, say $\mathbf{v}(\mathbf{x}, t)$, where $\mathbf{x}$ indicates the position and t the time. The velocity field therefore assumes well-defined values in every point of space and time crossed by the flow of our fluid. The variations of the velocity field as $\mathbf{x}$ varies are related to its variations in time and to the forces by which these variations are generated. The speed of the molecules of the fluid can in fact be varied with appropriate actions (forces), caused by external or internal agents, for example by variations in temperature in certain regions of the fluid. The relationships between these variations or *transformations* of the field and the forces define the *equations of the velocity field* and these describe the *dynamics* of our fluid current, the evolution of its state as space and time vary. The field therefore describes the collective motion of the fluid while being sensible point by point, locally, to the variations in the motion of the individual molecules.

In the following, instead of referring to the example of the fluid current and the velocity

field, a generic system and a generic field $\varphi(\mathbf{x}, t)$ will be considered.

We are interested in *continuous* transformations, i.e. those that depend on quantities, called parameters of the transformation, which vary continuously in a certain interval. For example, in the translations and rotations, the quantities of which the field is translated or the angle of which it is rotated vary continuously in a given interval.

The set of transformations of a certain type to which a field $\varphi(\mathbf{x}, t)$ can undergo, may enjoy well-defined mathematical properties and in this case it is said that it forms a *group of transformations*.

3. Symmetries and dynamics

It may happen that the equations of the fields do not change their mathematical form when the fields are transformed according to a group G of transformations. The equations, and therefore the dynamics they describe, are then said to be *symmetrical under the group G of transformations*.

Knowing the symmetries of dynamics is of great help in finding the solutions of the field equations. These describe the set of interactions between the elementary components and between these and the forces that operate. They are equations in which products and powers of the fields appear. For this reason they are called non-linear equations and finding their solutions can be very difficult.

The knowledge of symmetry properties offers the great advantage of being able to identify those solutions for which *conservation laws* apply, for example the conservation of energy and other quantities that characterize the states of the system. In fact, Noether's theorem[4] ensures that the existence of a continuous symmetry of the equations implies the existence of a corresponding quantity that does not vary with time (conserved in time).

4. The boundary conditions

There is another actor who has entered our story: the *state* of the system.

The systems we are interested in are generally composed by an enormous number of elementary components and the tools useful for their study are provided by quantum field theory (QFT). A central problem is the derivation of macroscopic properties and behaviors starting from the microscopic dynamics of quantum fields.

In QFT, the fields actually indicate mathematical operations (they are called indeed field "operators") which are well defined only on specific sets of functions, or *spaces of the states* of the system, called in the jargon of QFT *phases* or *representations* of the canonical field algebra.

A characteristic aspect of QFT is the existence of a set $\{H_F\}$ of an infinite number of possible different representations. They describe physically different realizations of the dynamics and are characterized by different values of quantities, the *order parameters*, relating to the symmetries of the dynamics. A specific space in the set of spaces $\{H_F\}$ depends on the specific properties of the environment in which the system evolves and with which it is *inextricably linked* (*entangled*, in the QFT jargon).

The *same* dynamics, *i.e.* the *same* set of field equations, governs the evolution of the states of the system in each of the different phases it can access.

Therefore, the assignment of the equations is not enough for the complete definition of the mathematical problem of the resolution of the field equations. It is necessary to specify also in which representation or phase one wants to solve them. The equalities between the members of the equations of the fields thus take on a defined mathematical meaning only when we operate with the fields on the states of the specified representation. This is expressed by saying that they are "weak equalities."

The transitions from one phase to another (*phase transitions*) are described by *critical* processes, that is, characterized by the unlimited growth (divergence) of certain specific quantities of the system.

A first conclusion we reach is therefore that the *same dynamics unfolds in a multiplicity of different physical phases or behaviors of the system*.

5. Interacting fields and asymptotic fields

The fields whose equations describe the interactions are called interacting or Heisenberg fields. Those in terms of which the observations are described are called asymptotic or physical fields. Let's denote the Heisenberg fields with ψ (x, t) and the asymptotic fields with φ ($\mathbf{x}$, t). H_H and H_F denote the spaces of the states on which ψ (x, t) and φ ($\mathbf{x}$, t) are respectively defined.

In general, observations are not possible in the spatial and temporal region in which the interaction takes place. Observations can in fact produce interference with the process to be studied. To avoid these interferences it is necessary to proceed with the observations in space-time regions far from the interaction region, in "asymptotic" regions. Only in these regions can we carry out the measurement operations of the observable quantities in terms of the fields φ ($\mathbf{x}$, t).

We therefore have no direct access to the interactions described by the dynamics. They remain *opaque*[5] to observations.

QFT therefore develops on two levels of language, that of the dynamics of the fields $\psi(\mathbf{x}, t)$ and the phenomenological one of the fields $\varphi(\mathbf{x}, t)$ (Figure 1). The connection between the two levels is given by the "dynamic map" Ψ which expresses ψ ($\mathbf{x}$, t) in terms of $\varphi(\mathbf{x}, t)$: $<\psi(\mathbf{x}, t)> = <\Psi(\varphi(\mathbf{x}, t))>$; the symbol < > denotes that the value of $\psi(\mathbf{x}, t)$ on the asymptotic states is obtained by operating with $\Psi(\varphi(\mathbf{x}, t))$ on them. Equality holds "in a weak sense" in the H_F space (see Section 4).

The functional form of Ψ in terms of $\varphi(\mathbf{x}, t)$ contains all the information contained in the equations of the dynamics; the map Ψ: $\psi(\mathbf{x}, t) \leftrightarrow \varphi(\mathbf{x}, t)$ is therefore called dynamical map.

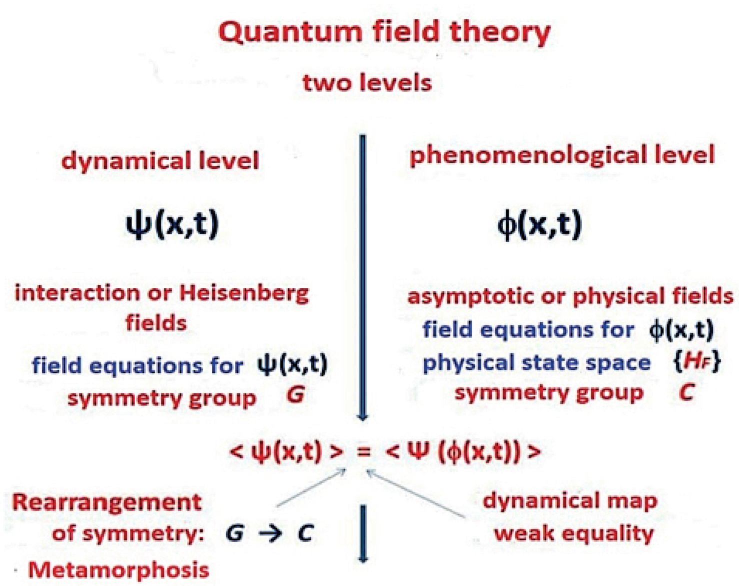

Fig. 1. The two levels of quantum field theory. When the space of physical states H_F is not symmetrical under the symmetry group G of the dynamics, but under the group C different from G, there is spontaneous breakdown of symmetry and the dynamical rearrangement G → C occurs.

6. Spontaneous breakdown of symmetry and generation of ordered structures

In addition to the symmetry properties of the Heisenberg field equations under the transformation group G, it is also necessary to consider the symmetry properties of the physical state spaces in the set $\{H_F\}$ in which the dynamics can be realized. It may happen that one or some of these spaces do not have the same symmetry properties of the dynamical Heisenberg equations.

Consider a specific H_F and its state of minimum energy, called the *vacuum* state. Suppose this state is symmetric under a group of transformations C that is different from G (Figure 1). When this happens, *spontaneous breakdown of symmetry* (SBS) occurs (we are not interested here in the *explicit* symmetry breaking obtained by modifying the equations of the fields with additional terms).

When SBS occurs, the equations for the asymptotic fields $\varphi(\mathbf{x}, t)$ defined on H_F are symmetric under C and there exists an observable quantity, let's denote it with M, distinctive of the vacuum state and of the considered H_F space, called *order parameter*. The reason for this name for M lies in the fact that SBS generates dynamic correlations between elementary components over large distances relative to their size. Such correlations are responsible for the formation of ordered structures in the states of the system. The order parameter provides a measure of the degree of ordering in the observable states.

Order therefore arises from the breaking of symmetry, it is *lack of symmetry*. A system that is symmetrical under certain transformations is in fact, by definition of symmetry, a system that remains unchanged even after it has undergone the transformations. The presence of symmetries produces a condition of indistinguishability between the system states before and after the transformation has been induced. The breaking of symmetry introduces the possibility of distinguishing between aspects or elements of the system that are otherwise indistinguishable.

For example, in a gas of atoms, each of them can be placed in any position; the dynamics is symmetrical under continuous spatial translation (original symmetry group G). Now suppose that the boundary conditions (for example variations in temperature, pressure, *etc.*) induce the gas to transform into a crystal (transition from the gaseous to the crystalline phase). In the crystal, the atoms are arranged in the sites of the crystal lattice and cannot be translated at will, as was the case in gas. The crystalline order thus arises from the breaking of continuous symmetry under space translation. The transformation into a crystal is a process called *dynamical rearrangement of symmetry.*[6]

Responsible for the ordering of the atoms are the correlations between them, *collective modes* or waves extending over the entire crystal: the elastic waves, whose associated quanta are the *phonons*.

The order parameter is given by the density of the crystal, linked to the number of phonons *condensed* in the ground state. This number can be varied with a condensation transformation.

In conclusion, the *crystalline form* is generated from the gas of atoms in a process of dynamic transformation. SBS thus complement the study of the system components (the atomist standpoint) with their collective dynamical interactions (the dynamical standpoint). *Naturalism*, limited to the atomist vision, is necessary but not sufficient. Inclusion of the dynamical vision leads to *scientific knowledge*.

Another example, among many, is the magnet. At the level of the basic dynamics, the elementary magnets (e.g. the electrons or the atoms with a magnetic moment) can each be oriented in any direction. The group G is that of continuous spherical rotations. Following the SBS, G is transformed (*rearranged*) into the group C that contains the cylindrical rotations around the specific direction of the magnetization (order parameter) and the condensation of *magnons*, quanta of the correlation waves (spin waves) between the elementary magnets. The order parameter thus characterizes the macroscopic *form* of the system, namely its macroscopic behavior as a magnetized system.

In these two examples, variations of the boundary conditions, *e.g.* of the temperature, induce variations of the order parameter (density and magnetization, respectively), namely transformations in the crystalline and magnetic structures, respectively.

The system thus goes, through these transformations, from *form to form*. The basic dynamics in each of the cases *manifests itself* at the level of observations in a multiplicity of different orders, different *forms*: *meta-morphosis* from the opacity of the original uniformity (symmetry) to the richness of diversity.

7. Dissipation, coherence and the arrow of time

In solving the Heisenberg field equations the assignment of a specific state space H_F corresponds to considering the properties of the environment. The *dissipative* character of the dynamics of the system is thus considered. The {system-environment} complex constitutes a single *closed* system and the flows of the exchanges between them are balanced. This closing operation is necessary since the available mathematical formalism (called canonical) is modeled for closed systems.

The metamorphosis process originates precisely in the realization of the dynamics in H_F. Therefore the dissipative character of the dynamics plays an essential role in the generation of forms (morphogenesis) to which the rearrangement of the symmetry leads.

Since variations in the boundary conditions are also induced by interactions with the environment, and since these variations induce phase transitions (metamorphoses), from H_F to H_F', H_F'' and so on, we see that the "history" of the system evolves through trajectories in $\{H_F\}$ in a succession of phase transitions in its interaction with the environment.

The reorganization (rearrangement) of symmetry is a *dynamic* process. The *transformation from form to form* in the succession of *metamorphoses* does not consist in the *negation* of the basic symmetry of the field equations, but in its *disclosure* through the richness of *possible, different modalities of existence*.

For the sake of brevity, I do not dwell further on phase transitions, although they play a fundamental role; for example, a continuous succession of phase transitions characterizes the evolution over time in biological systems, neuroscience,[7] and functional specialization in general. However, it should be emphasized that on the trajectory from space to space (metamorphoses), the minimization of free energy is ensured in each of the spaces through which it proceeds. This ensures that although the system evolves through a continuum of phase transitions, it is stable in each H_F (the system is "locally" stable). A property that guarantees the functional stability of the system.

Minimization of free energy implies the energy balance linked to the formation of ordered structures and therefore to the entropy and *irreversible time evolution* of the system. The dissipative character of the dynamics ultimately implies that the system cannot evolve going backward in time, *i.e.* breaking of time reversal symmetry, the appearance of *the arrow of time*: metamorphoses are not reversible (perhaps it is not a case that in fairy tales and myths undoing a metamorphosis, i. e. breaking a spell, requires a miraculous action… only the kiss of the princess can reverse the arrow of time by returning the frog to what it was before, a beautiful prince).

The energy transferred to an ordered system is distributed not only among the elementary components, but also to the "network of correlations" that binds them in the ordering. In a disordered system, for example in a gas, the acquired energy is distributed among the elementary components producing, apart from their transition to excited states when such a possibility exists, an increase in kinetic energy (thermalization with heat production and diffusion as predicted by the kinetic theory of gases). In ordered systems, the presence of the correlation waves, represented by their associated quanta, imposes the distribution of energy also to the correlation network itself. This entails a reduced thermalization and the possibility of collecting energy in the system, "keeping it on the correlation network" for the purpose of subsequent use (in chemical reactions or other) inside the system or in its interactions with the environment.

To better understand how energy, dissipation and (local) stability are linked to the formation of ordered structures, it should be remembered

that the phenomenon of condensation, induced by SBS, is described by the transformation of the correlation quanta $B(\mathbf{x}, t) \rightarrow B(\mathbf{x}, t) + c(\mathbf{x}, t)$, with c dependent or not on **x** and t (non-homogeneous or homogeneous condensation, respectively).

The condensation transformation of $B(\mathbf{x}, t)$ produces states characterized by the fact that the correlations they represent do not interfere destructively because they are "in phase" with each other, they are *coherent states*.

The coherence property allows the possibility of the transition from the microscopic (quantum) world to the macroscopic (classical) behaviors of the system. This is possible since in coherent states the quantum fluctuations $\langle \Delta N \rangle$ of the number $\langle N \rangle$ of the condensed quanta are negligible in percentage; in fact, $\langle \Delta N \rangle / \langle N \rangle \approx 1/|\alpha|$, where $|\alpha|$ denotes the degree of coherence of the state, so that greater $|\alpha|$ (coherence) implies lower percent of quantum fluctuations and the system therefore shows classical behaviors. The order parameter is in fact a classical field in the sense that its value does not depend on quantum fluctuations, and this precisely indicates the stability (with respect to quantum fluctuations) of the ordering it accounts for. It is in this sense that we refer to systems that present ordering as *macroscopic quantum systems*.

In conclusion, the set $\{H_F\}$ of the state spaces of the system is a set of coherent states and it can be shown that the trajectories through which the system evolves, from phase to phase, are classic chaotic trajectories, i.e. such that small variations in the initial conditions imply divergent trajectories, never wrapping around themselves. The system is therefore able to discriminate between small variations of the initial conditions, resulting in different behaviors. The properties of chaos give the system great functional efficiency.

The phenomenon of coherence is thus at the basis of the metamorphoses through which the underlying dynamics manifests itself. Particularly notable is the case of fractals or self-similar structures discussed in the next section.

8. Fractals and coherence

Consider the Koch curve (Figure 2) as an example of fractal curve. Divide the segment u_0 into 3 parts ($\lambda = 1/3$). With 4 of these segments, each equal to $u_0/3$, construct the segment $u_1 = (4/3)\ u_0$. Put p = 4. Impose $u_1/u_0 = 1$, which amounts to asking that the path u_0 and u_1 be equivalent paths (there are interesting processes in physics that do not depend on the path followed in passing from point A to point B). Therefore, in order to satisfy $u_1/u_0 = 1$, there must be a number d such that $4/3^d = \lambda^d p = 1$. Repeating the process *n* times, for every integer *n*, with $n \rightarrow \infty$, we have $(4/3^d)^n = (\lambda^d p)^n = 1$. Thus, d = log 4/log 3 = 1.2619.

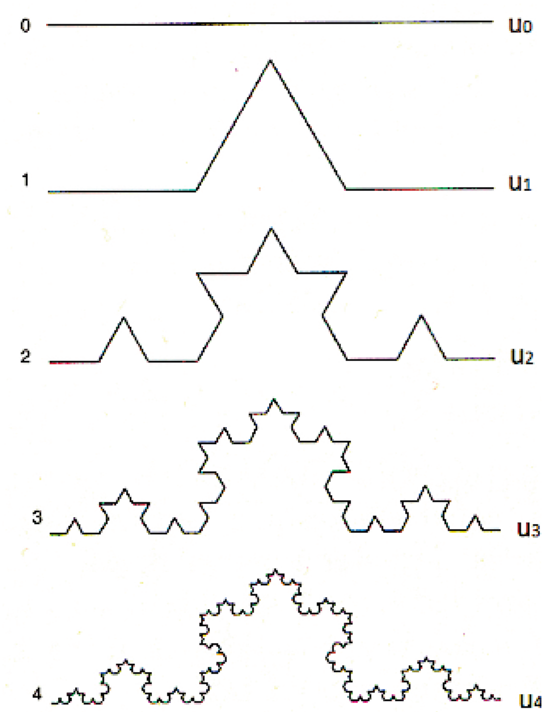

Fig. 2. The first 5 stages of the Koch curve.

This number d defines the fractal dimension, or self-similarity dimension, of the Koch curve. When λ and p are extended to complex values, the quantities $(\lambda^d\ p)^n$, for each integer *n*, apart a normalization factor, constitute the functions by which coherent states are constructed in quantum theories. One can prove then that fractals and deformed coherent states are mathematically isomorphic.[8] Fractals can thus be thought as macroscopic quantum systems resulting from the deformation of the coherent microscopic dynamics. As the values assumed by the deformation parameter $q = \lambda^d$ vary, the dynamics of the elementary components of the system manifest itself in different fractal forms. Coherence, that is, the harmonious fabric of long range correlations between elementary components, generates the multiplicity of self-similar structures that we observe in nature.

9. Origin of the metamorphoses

It is possible to demonstrate[9] that the process of selection of the symmetry breaking state can be induced by a weak stimulus (trigger), a minimal but in-phase input, *i.e.* capable of resonating with the system. Such a process of symmetry breaking via a minimum input turns out to be linked to the localization of physical states within finite spatial and temporal boundaries. Their local nature is ultimately at the origin of the dynamical rearrangement of symmetry, *i.e.* of metamorphoses.

As matter of fact, our observations are always local. The local nature is an intrinsic aspect to them, which we cannot avoid, and which reflects on the observed physical states. The spatial-temporal limitations induced by the localization are however essential because otherwise, we could not distinguish "what from what." For example, the possibility of talking about an apple arises from the fact that in our observations we can localize it, that is, its defined spatial contours avoid its "overlapping" to an orange… In short, our observations always depend on a "threshold" volume V. Contributions from what happens outside V are of the order of 1/V, that is, they are negligible for large V ($1/V \rightarrow 0$ as V increases). The dynamical rearrangement of symmetry finds its origin in the missing of these contributions of the order of 1/V in our observations. If, on the other hand, we recover the contributions 1/V by resuming them and take them into account in our analysis, we can (mathematically) trace the rearrangement path and "recognize" (within certain limits) the invariance group G of the equations of the interaction fields.[10]

10. Conclusions

Fabrizio Desideri[11] recalls that in Plato's Cratilo "*beauty is* (…) *understood as the eponym of dianoia. Therefore, it does not express the stability of a thing, but the dynamics of an activity, that of naming. In to kalòn, therefore, the denominative power of intelligence resounds: its ability to establish names and, thus, to be able to call entities.*" This passage, as already observed elsewhere,[12] offers me the possibility of noting that "establishing names" and "being able to call entities", that is, to distinguish them from one another, introduces the spontaneous breakdown of the symmetry corresponding to their indistinguishability existing before each is given a name.

Thus we see how general the process of symmetry breaking can be. In linguistics, for example, we can imagine having a set of letters, with symmetry under permutation allowing them to be interchanged. Suppose we choose four of them for simplicity, *e.g.* r, m, a, o. The symmetry under permutation can be broken by choosing to align them, for example, in the order "roma," which, in Italian, denotes the city of Rome. A word corresponding to a different order (a different $H_{r'}$ in the notation of the previous Sections) could be "orma." Similarly, we could get "amor," "omar," "ramo," *etc.* All different orderings (different phases for our system of four letters), different *forms* originating from the reorganization of the symmetry under permutations (metamorphosis). In the Italian language (environment), each of these words is related to other words in a network of correlations over "distances" greater than those on which the elementary components live and relate. This network of correlations defines the *meanings*: "orma" is the footprint left in the sand, "omar" is our friend, *etc.* Different meanings associated with different orderings. The meaning of orma therefore does not belong to the "r" or "m," or any of the other component letters, but to their specific ordering, it is "shared" by them, it is the *collective mode* (coherence) that "wraps" the four letters in their correlation and with other words in the specific linguistic and cultural context. The result is the *dynamic* passage from the level of elementary components to that of meanings, from syntax to semantics.

In addition to the simple example of letters in constructing words, we can consider the next level of ordering between words in forming sentences, and so on in levels of greater complexity.[13]

The discussion on SBS, dynamical rearrangement of symmetry (the dynamical process of metamorphosis) can be extended to the study of the brain functional activity and to biology in general. For the sake of brevity, I do not report on it here.[14]

In conclusion, the mathematical structure of QFT shows that the symmetry of the basic interaction dynamics, not accessible to our observations, manifests itself at the macroscopic level in a multiplicity of ordered patterns through metamorphosis processes ruled by the coherence paradigm. We could say that in QFT the focus is on the study of the metamorphosis of the formless uniqueness of *being* in the multiple diversity of the *existing*. Perhaps it is appropriate to conclude with Darwin's words about the generation and evolution of forms in living matter:

> "*There is grandeur in this view of life, with its several powers, having been originally breathed into a few forms or into one; and that, whilst this planet has gone cycling on according to the fixed law of gravity, from so simple a beginning endless forms most beautiful and most wonderful have been, and are being, evolved.*"[15]

[1] I am grateful to F. Desideri, U. Fadini and P. F. Pieri for allowing me to report here the translation of the paper on a similar subject published in the magazine *Atque*. Cf. G. Vitiello, "Simmetrie e Metamorfosi," *Atque 24 n.s.*, 2019, 139-160. I also thank Laetitia D'Elia for illuminating discussions.

[2] Dipartimento di Fisica "E.R. Caianiello," Università di Salerno, Italia.

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Localité et globalité

Jean-Paul Allouche[1]

Alors que l'emploi des mots « local » et « global » est de plus en plus fréquent, nous nous interrogeons sur les raisons explicites ou cachées qui favorisent l'usage de l'un ou l'autre de ces concepts.

Quel lien existe-t-il entre les notions de localité et de globalité : complémentarité ou antonymie ? Le premier champ que nous questionnons est celui des mathématiques, où, à défaut d'être toujours explicitement définies, elles ont souvent un sens précis pour ceux qui les utilisent ou les manient. Ce qui semble le plus frappant est que le passage en douceur de l'un à l'autre de ces deux mots cache souvent des théorèmes dont la démonstration est loin d'être immédiate, comme le suggèrent peut-être les trois exemples qui suivent. Le premier résultat de ce type est le fait que *le corps C est algébriquement clos*. Que signifie cet énoncé ? On apprend au collège ou au lycée qu'un carré est toujours positif ou nul, donc, en particulier, que -1 n'est pas un carré, autrement dit que l'équation polynomiale $x^2 + 1 = 0$ n'a pas de racine. Les mathématiciens, qui aiment bien transgresser les règles pour en établir de plus générales, ont décidé (la réalité historique est un peu différente) de « forcer » cette équation à avoir une solution, notée « i », et de voir où cela pourrait mener. Sans surprise, et en tentant de respecter autant que possible les règles habituelles malgré l'incongruité d'avoir maintenant une racine carrée du nombre -1, ils constatent (démontrent) qu'avec le nouveau nombre i apparaissent « l'autre » racine carrée de -1, a savoir -i, puis tous les nombres de la forme a + bi où a et b sont des nombres réels (autrement dit les nombres que l'on manipule habituellement comme 2, -3/4, $\sqrt{2}$, *etc.*). Puis ils constatent que les opérations habituelles menées sur ces nombres n'en créent pas de nouveaux : l'ensemble de ces nombres est appelé le *corps des nombres complexes* (le mot *corps* signifiant que l'on peut mener des calculs avec les quatre opérations comme d'habitude, sans oublier l'interdiction de diviser par zéro ; le mot *complexe*, lui, n'étant pas là pour décourager les Béotiens, mais seulement (?) pour signifier qu'il ne s'agit pas des nombres ordinaires). En un certain sens, cet ensemble de nombres semble conçu pour résoudre un problème localisé, à savoir permettre que le polynôme $x^2 + 1$ ait une racine (à savoir i). Or ceci implique non seulement que ce polynôme a maintenant deux racines (i et -i), mais encore – et c'est là un résultat superbe appelé parfois théorème fondamental de l'algèbre – que tout polynôme de degré d à coefficients réels a d racines distinctes ou partiellement confondues dans le corps C, et même mieux : ceci est vrai pour tout polynôme à coefficients complexes (on dit donc que C est *algébriquement clos*). Ainsi un modeste ajout (celui de i) qu'on peut voir comme la résolution d'une question très

localisée entraîne-t-il une propriété globale : on obtient *ipso facto* et sans l'avoir voulu une explosion de cette propriété qui s'étend de manière globale bien plus loin que la question fondatrice.

Un deuxième exemple est la construction des nombres réels (comme $\sqrt{2}$) à partir des nombres rationnels (les quotients de nombres entiers). Imaginons qu'on veuille calculer un nombre dont le carré vaut 2 (noté $\sqrt{2}$) par approximations successives, soit par un algorithme itératif « efficace » fondé sur la méthode de Newton, soit par un procédé qu'on n'apprend plus guère à l'école (où l'on a une « potence » semblable à celle qu'on utilise pour la division, *etc.*). On construit alors des approximations rationnelles par défaut et par excès de $\sqrt{2}$, par exemple :

$$1 < \sqrt{2} < 2$$
$$1,4 < \sqrt{2} < 1,5$$
$$1,41 < \sqrt{2} < 1,42$$
$$1,414 < \sqrt{2} < 1,415$$
$$\ldots$$

Les encadrements écrits signifient que s'il existe un nombre qui a 2 pour carré, il est compris entre 1 et 2, puis entre 1,4 et 1,5, puis entre 1,41 et 1,42, *etc.* Ces deux suites de rationnels semblent *converger* vers (se rapprocher de plus en plus de) « quelque chose » qui n'est pas un nombre rationnel (ce qu'on sait depuis les textes d'Aristote). Mais vers quoi convergent-elles ? Soucieux de précision, les mathématiciens (là encore je simplifie la réalité historique) constatent que la suite 1 1,4 1,41 1,414... a la propriété que ses termes sont tous arbitrairement proches les uns des autres dès que leurs indices sont suffisamment grands : on peut aussi formuler cette propriété en disant que si on se donne un intervalle, aussi petit que l'on veut, tous les termes de la suite y sont enfermés dès que leurs indices sont assez grands. De telles suites sont appelées *suites de Cauchy*. Notons que toute suite *qui tend vers une limite* (par exemple 1 1,1 1,11 1,111… qui tend vers 10/9, c'est-à-dire qui s'approche de plus en plus de la valeur 10/9) est une suite de Cauchy, mais que la réciproque n'est pas vraie, comme le montre l'exemple des approximations 1,414... ci-dessus. Que faire alors ? Les mathématiciens *décident* que les suites de Cauchy *sont* de nouveaux nombres (à l'identification près de deux telles suites lorsque leur différence tend vers 0, c'est-à-dire devient aussi proche de 0 que l'on veut pourvu que l'on prenne des indices assez grands) et obtiennent ainsi (après démonstration rigoureuse) un ensemble de *nouveaux nombres*, qui contient les rationnels et tous les nombres que l'on peut obtenir (comme $\sqrt{2}$) par ce procédé appliqué aux nombres rationnels. Et cette résolution d'une question somme toute locale, explose, car on montre ensuite que les suites de Cauchy fabriquées à partir de ces nouveaux nombres ne donnent rien de nouveau si on leur applique le même procédé. Une sorte de « saturation » inattendue *a priori* permet d'obtenir plus que ce qu'on espérait.

Un dernier exemple, avant de clore cette partie aux allusions mathématiques, est celui des fonctions *holomorphes*. Une courbe *continue* (la signification intuitive de cette notion est qu'on peut tracer la courbe sans lever le crayon de la feuille) peut admettre des tangentes en certains points (une tangente en un point est obtenue en prenant une sécante entre ce point et un autre point de la courbe et en faisant tendre ce deuxième point vers le premier). Lorsque la courbe est la représentation d'une fonction, la pente de la tangente en un point est la dérivée de la fonction en ce point (c'est-à-dire la limite du rapport entre l'accroissement de la fonction et celui de la variable). Tangente et dérivée peuvent ne pas exister (penser aux points où une ligne brisée… se brise). Il y a même des fonctions longtemps considérées comme monstrueuses qui sont continues et nulle part dérivables. Le fait pour une fonction d'être dérivable en un point est *local* et ne dépend que des propriétés de la fonction au voisinage de ce point. Maintenant, si une fonction est dérivable sur un voisinage d'un point, on peut se demander si la dérivée est elle-même dérivable, puis si cette dernière dérivée (dérivée *seconde*) est elle-même dérivable et ainsi de suite, *ad infinitum*. Bien sûr, il n'y a aucune raison qu'une dérivée soit dérivable,